These problems will result in a *system* of differential equations.

Differential Equations
Class Notes
Introduction to Systems: Interconnected Fluid Tanks (Section 5.1)

Think back to our problems involving a single closed tank with brine circulating within. We will now add a second tank that is interconnected to the first. Here is a picture.

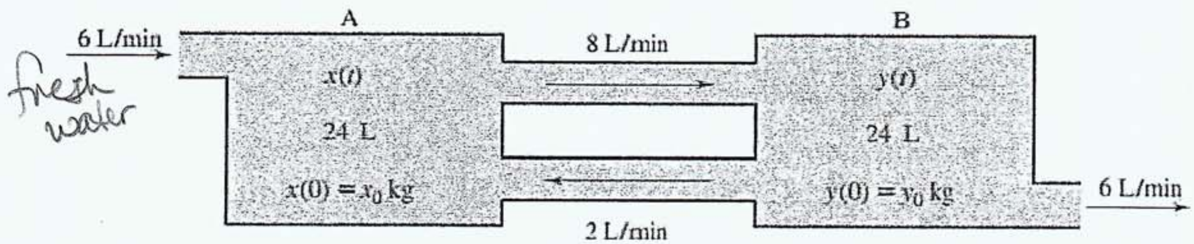

Figure 5.1 Interconnected fluid tanks

What we see here are two tanks, each with volume 24 liters (L). Tank A has $x(t)$ kilograms (kg) of salt dissolved in the water at time t minutes; Tank B has $y(t)$ kilograms (kg) of salt dissolved in the water at time t minutes. We will take t to be greater than zero. Fresh water enters tank A at the rate of 6 liters per minute on the left; briny water leaves tank B at the rate of 6 liters per minute on the right.

The tanks are interconnected. Briny water flows from tank A to tank B at a rate of 8 liters per minute; Briny water flows from tank B to tank A at a rate of 2 liters per minute. The liquids are kept well stirred so we assume they are homogeneous.

The initial *amounts* (kg) of salt in tanks A and B are, respectively, $x(0) = x_0$ and $y(0) = y_0$.

Since the system is being flushed with fresh water, it is true that as $t \to \infty$, both x and y diminish to zero.

(3.2)

Let's examine the flows and concentrations of salt more closely. As we have seen in a previous section, the salt *concentration* (kg/L) in tank A is $\frac{x(t)\ kg}{24\ L}$. The upper pipe carries salt *out* of tank A at a rate of $\frac{8\ L}{1\ \text{min}} \cdot \frac{x\ kg}{24\ L} = \frac{x\ kg}{3\ \text{min}}$. Similarly, the salt *concentration* (kg/L) in tank B is $\frac{y(t)\ kg}{24\ L}$. The lower pipe carries salt *into* tank A at a rate of $\frac{2\ L}{1\ \text{min}} \cdot \frac{y\ kg}{24\ L} = \frac{y\ kg}{12\ \text{min}}$.

This gives us $\dfrac{dx}{dt} = input\ rate - output\ rate = \dfrac{y}{12} - \dfrac{x}{3}$. Similarly, for tank B, we have

$$\frac{dy}{dt} = input\ rate - output\ rate$$
$$= \frac{x}{3} - \left(\frac{2y}{24} + \frac{6y}{24}\right)$$
$$= \frac{x}{3} - \frac{y}{3}$$

Tank B has two outlet pipes.

Look at that! We have a system of equations.

$$x' = -\frac{1}{3}x + \frac{1}{12}y$$
$$y' = \frac{1}{3}x - \frac{1}{3}y \rightarrow 3y' = x - y$$
$$3y' + y = x$$

Solve the second equation for x and substitute it into the first equation. This yields

$$x' = -\frac{1}{3}x + \frac{1}{12}y$$
$$(3y' + y)' = -\frac{1}{3}(3y' + y) + \frac{1}{12}y$$
$$3y'' + y' = -y' - \frac{1}{4}y$$
$$3y'' + 2y' + \frac{1}{4}y = 0$$

This is a second-order, linear diff. eq. with constant coefficients.

As we have seen before, this can be solved by solving its auxiliary equation, $3r^2 + 2r + 1/4 = 0$.

We get $r = -1/2, -1/6$. That gives us a general solution of $y(t) = c_1e^{-1/2} + c_2e^{-1/6}$ where c_1 and c_2 are real numbers. Complete the process to find c_1 and c_2.

Here, find y' and then x given that $x = 3y' + y$.

$$\boxed{y = c_1 e^{-t/2} + c_2 e^{-t/6}}$$
$$y' = -\frac{1}{2}c_1 e^{-t/2} - \frac{1}{6}c_2 e^{-t/6}$$

$$x = 3y' + y$$
$$= -\frac{3}{2}c_1 e^{-t/2} - \frac{1}{2}c_2 e^{-t/6} + c_1 e^{-t/2} + c_2 e^{-t/6}$$
$$\boxed{x = -\frac{1}{2}c_1 e^{-t/2} + \frac{1}{2}c_2 e^{-t/6}}$$

To finish this out, solve for c_1 and c_2 by recalling $x(0) = x_0$ and $y(0) = y_0$. This yields a system of equations involving c_1 and c_2. This process will be generalized in the next section to find solutions of all linear systems with constant coefficients.

$$y = c_1 e^{-t/2} + c_2 e^{-t/6}$$
$$x = -\tfrac{1}{2} c_1 e^{-t/2} + \tfrac{1}{2} c_2 e^{-t/6}$$

$x(0)=x_0$: $\quad x_0 = -\frac{1}{2} c_1 \cancel{e^0}^{1} + \frac{1}{2} c_2 \cancel{e^0}^{1}$

$y(0)=y_0$: $\quad y_0 = c_1 + c_2$

$$2x_0 = -c_1 + c_2$$
$$y_0 = c_1 + c_2$$

add

$$2x_0 + y_0 = 2c_2$$
$$c_2 = x_0 + \tfrac{1}{2} y_0$$

$$y_0 = c_1 + c_2$$
$$y_0 = c_1 + x_0 + \tfrac{1}{2} y_0$$
$$\tfrac{1}{2} y_0 - x_0 = c_1$$

Soln: $\quad y = \left(\frac{1}{2} y_0 - x_0\right) e^{-t/2} + \left(x_0 + \frac{1}{2} y_0\right) e^{-t/6}$

$$x = -\tfrac{1}{2}\left(\tfrac{1}{2} y_0 - x_0\right) e^{-t/2} + \tfrac{1}{2}\left(x_0 + \tfrac{1}{2} y_0\right) e^{-t/6}$$

$$x = \left(-\tfrac{1}{4} y_0 + \tfrac{1}{2} x_0\right) e^{-t/2} + \left(\tfrac{1}{2} x_0 + \tfrac{1}{4} y_0\right) e^{-t/6}$$